

Coloured Hard-Dimers

Maria Simonetta Bernabei
and
Horst Thaler

Department of Mathematics and Informatics,
University of Camerino,
Via Madonna delle Carceri 9, I-62032, Camerino (MC), Italy;
simona.bernabei@unicam.it, horst.thaler@unicam.it

Abstract

An averaged generating function for coloured hard-dimers is being investigated by proving estimates for the latter. Furthermore, two different enumerating problems and their distributions are studied numerically.

1 Introduction

Coloured hard-dimers have recently appeared in the context of causally triangulated $(2+1)$ -dimensional gravity. Using special triangulations of spacetime, it was shown in [1] that the generating function of the one step propagator can as well be expressed through the generating functions of coloured hard-dimer configurations.

In this article we go in a slightly different direction and study the behaviour of an averaged generating function for coloured hard-dimers. In this manner we touch some combinatorial problems, which so far have not been tackled.

The paper is organized as follows. In section 2, we define the notion of coloured hard-dimers, give the statement of the problem and prove estimates

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for the averaged generating function. In section 3 additional material for numerics around coloured hard-dimers is presented.

2 Description of the model and results

Let ξ_N be a sequence of length N of blue and red sites on the one-dimensional lattice \mathbb{Z} . Let us define the notion of a coloured hard-dimer on ξ_N : a coloured hard-dimer is a sequence of red or blue dimers, which cannot intersect each other (“hardness” property). A dimer in turn is an edge linking two nearest sites of the same colour. In Fig.1 an example of a coloured hard-dimer is given.



Figure 1: A hard-dimer, $N = 13$

For any configuration ξ_N , the generating function associated with it is of the form

$$Z_{\xi_N}(u, v, w) = \sum_D u^{n_b(D)} v^{n_r(D)} w^{n_{br}(D)}, \quad u, v, w \in (0, \infty), \quad (2.1)$$

where D is an hard-dimer on ξ_N , $n_b(D)$ and $n_r(D)$ denote the number of blue and red dimers respectively on D , and $n_{br}(D)$ the number of crossings between dimers and sites of different colour. Moreover, let us denote by $\gamma_b(D)$ and $\gamma_r(D)$ the number of blue and red sites of D respectively, which do not belong to any dimer. We call such kind of sites “single points”. Naturally the following constraint

$$2n_b(D) + 2n_r(D) + n_{br}(D) + \gamma_b(D) + \gamma_r(D) = N \quad (2.2)$$

holds. It is useful to define further variables t and s , where t corresponds to the number of sites occupied by dimers, and s to the number of dimers:

$$t = N - \gamma_b(D) - \gamma_r(D), \quad s = n_b(D) + n_r(D) \quad (2.3)$$

In the above example $n_b(D) = 1$, $n_r(D) = 2$, $n_{br}(D) = 3$, $\gamma_b(D) = 3$ and $\gamma_r(D) = 1$.

For any N , we want to prove estimates from above and from below, for the mean of the generating functions Z_{ξ_N} defined in (2.1). In achieving this we calculate explicitly, for a given N , the sum of all the generating functions Z_{ξ_N} over ξ_N , rescaled by the factor 2^N , i.e. the number of all the sequences ξ_N . We get so an “averaged” generating function

$$\frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}(u, v, w).$$

The main result of this section is the following:

Theorem 2.1 *For any N , the following estimates*

$$1 - \frac{C_1}{2^N \sqrt{N}} + \frac{C}{\sqrt{N}} \left(\frac{A+1}{2} \right)^N \leq \frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}(u, v, w) \leq 1 - \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} (A+1)^N \quad (2.4)$$

hold, where $A \equiv \sqrt{u+v} + \frac{w}{2}$ and C_1, C are constants not depending on N , given as

$$C_1 = e^{-\frac{1}{6}} \sqrt{\frac{2}{\pi}}, \quad C = \frac{C_1}{2} \left(1 \wedge \frac{w+2}{\sqrt{u+v} + \frac{w}{2} + 1} \right)$$

Remark 2.1 *Note that $A > 0$, so that the upper bound in (2.4) is always exponentially divergent, as $N \rightarrow \infty$, whereas the lower bound increases exponentially for $A > 1$ and large N , i.e. for the class of parameters u, v, w , such that $\sqrt{u+v} + \frac{w}{2} > 1$, otherwise it tends to 1.*

In order to prove the main result, we first need a Lemma, where we get an explicit formula for the average $\frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}$, for any N . We are able to get an exact formula by summing over the variables t and s defined above in (2.3).

Lemma 2.1 *The average $\frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}$ has the following explicit expression, for any N :*

$$\begin{aligned} & \frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}(u, v, w) = \\ & 1 + \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \binom{N-t+s}{s} \binom{t-s-1}{s-1} \left(\frac{u+v}{4} \right)^s \left(\frac{w}{2} \right)^{t-2s} \end{aligned} \quad (2.5)$$

where $\lfloor \cdot \rfloor$ denotes the integer part.

Proof of Lemma 2.1: Consider a hard-dimer D of length N , and set $n_b(D) = k$, $n_r(D) = h$, $n_{br}(D) = m$, $\gamma_b(D) = \gamma_b$ and $\gamma_r(D) = \gamma_r$. By (2.2) and (2.3) the following equalities

$$2k + 2h + m + \gamma_b + \gamma_r = N, \quad t = 2k + 2h + m \quad (2.6)$$

hold.

First we fix the number of blue and red dimers and that one of single points γ_b and γ_r , without assigning any length to dimers, even though, by (2.3), the variable t on D is done. In this manner the dimers of the same colour are considered indistinguishable. Then we calculate all possible permutations of k blue dimers, h red dimers, γ_b blue and γ_r red single points, i.e.

$$\frac{(k + h + \gamma_b + \gamma_r)!}{k! h! \gamma_b! \gamma_r!}$$

Now, one has to distinguish two cases: $k = h = 0$ and $k + h \neq 0$. If $k = h = 0$, then $m = 0$, because this means that there are no dimers. Therefore $\gamma_b + \gamma_r = N$ and the combinatorics is

$$\sum_{\gamma_b=0}^N \binom{N}{\gamma_b} = 2^N$$

In the other case ($k + h \neq 0$) we assign, for any given permutation of coloured dimers and single points, all the admissible lengths to each dimer, taking into account the fact that t is given. The number of all these combinations is then

$$\binom{m + k + h - 1}{k + h - 1}$$

Therefore we have

$$\begin{aligned} \sum_{\xi_N} Z_{\xi_N}(u, v, w) = \\ \sum_{\substack{k, h, m: \\ 2k + 2h + m = 1}}^N \sum_{\gamma_b + \gamma_r = N - 2k - 2h - m} \frac{(k + h + \gamma_b + \gamma_r)!}{k! h! \gamma_b! \gamma_r!} \binom{m + k + h - 1}{k + h - 1} u^k v^h w^m \end{aligned} \quad (2.7)$$

From (2.6) we have that $k + h + \gamma_b + \gamma_r = N - k - h - m$ and $\gamma_r = N - 2k - 2h - m - \gamma_b$ and hence

$$\sum_{\xi_N} Z_{\xi_N}(u, v, w) =$$

$$\begin{aligned}
& \sum_{\substack{k,h,m: \\ 2k+2h+m=1}}^N \sum_{\gamma_b=0}^{N-2k-2h-m} \frac{(N-k-h-m)!}{k! h! \gamma_b! (N-2k-2h-m-\gamma_b)!} \binom{m+k+h-1}{k+h-1} u^k v^h w^m = \\
& \sum_{\substack{k,h,m: \\ 2k+2h+m=1}}^N \frac{(N-k-h-m)!}{k! h! (N-2k-2h-m)!} \binom{m+k+h-1}{k+h-1} u^k v^h w^m \times \\
& \sum_{\gamma_b=0}^{N-2k-2h-m} \binom{N-2k-2h-m}{\gamma_b} \quad (2.8)
\end{aligned}$$

In (2.8) we have multiplied and divided the generic term of the sum by $(N-2k-2h-m)!$, so that the binomial formula appears

$$\sum_{\gamma_b=0}^{N-2k-2h-m} \binom{N-2k-2h-m}{\gamma_b} = 2^{N-2k-2h-m}$$

and $\sum_{\xi_N} Z_{\xi_N}$ becomes

$$2^N \left[1 + \sum_{\substack{k,h,m: \\ 2k+2h+m=1}}^N \frac{(N-k-h-m)!}{k! h! (N-2k-2h-m)!} \binom{m+k+h-1}{k+h-1} \left(\frac{u}{4}\right)^k \left(\frac{v}{4}\right)^h \left(\frac{w}{2}\right)^m \right] \quad (2.9)$$

Note that the scaling factor 2^N naturally appears in (2.9). Now, by multiplying and dividing the generic term of the sum in (2.9) by $(k+h)!$ we get

$$\begin{aligned}
& \frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}(u, v, w) = \\
& 1 + \sum_{\substack{k,h,m: \\ 2k+2h+m=1}}^N \binom{N-k-h-m}{k+h} \binom{m+k+h-1}{k+h-1} \binom{k+h}{k} \left(\frac{u}{4}\right)^k \left(\frac{v}{4}\right)^h \left(\frac{w}{2}\right)^m
\end{aligned}$$

We perform the variable changements $s = k + h$ and $t = 2k + 2h + m$, where s and t were already defined. We get thus

$$\begin{aligned}
& \frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}(u, v, w) = \\
& 1 + \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \binom{N-t+s}{s} \binom{t-s-1}{s-1} \left(\frac{w}{2}\right)^{t-2s} \sum_{k=0}^s \binom{s}{k} \left(\frac{u}{4}\right)^k \left(\frac{v}{4}\right)^{s-k} =
\end{aligned}$$

$$1 + \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \binom{N-t+s}{s} \binom{t-s-1}{s-1} \left(\frac{u+v}{4} \right)^s \left(\frac{w}{2} \right)^{t-2s} \quad (2.10)$$

In the last sum in (2.10) we used again the binomial formula

$$\sum_{k=0}^s \binom{s}{k} \left(\frac{u}{4} \right)^k \left(\frac{v}{4} \right)^{s-k} = \left(\frac{u+v}{4} \right)^s$$

Therefore we get formula (2.5) where only the indices s and t appear. The Lemma is so proved. \square

Proof of Theorem 2.1: First we write the sum on the right hand side of formula (2.10) in a more suitable form:

$$\binom{N-t+s}{s} \binom{t-s-1}{s-1} = \frac{s \binom{2s}{s}}{N \binom{N-1}{t-s-1}} \binom{N}{2s} \binom{N-2s}{t-2s}, \quad s \geq 1, t \geq 1 \quad (2.11)$$

In fact, it holds

$$\begin{aligned} & \binom{N-t+s}{s} \binom{t-s-1}{s-1} = \\ & \frac{(N-t+s)!}{s!(N-t)!} \cdot \frac{(t-s-1)!}{(s-1)!(t-2s)!} \cdot \frac{N!}{N!} \cdot \frac{(N-2s)!}{(N-2s)!} \cdot \frac{(2s)!}{(2s)!} = \\ & \frac{\frac{(2s)!}{s!(s-1)!}}{\frac{N!}{(N-t+s)!(t-s-1)!}} \binom{N}{2s} \binom{N-2s}{t-2s} = \\ & \frac{s \binom{2s}{s}}{N \binom{N-1}{t-s-1}} \binom{N}{2s} \binom{N-2s}{t-2s} \end{aligned}$$

Now we estimate the binomial coefficients $\binom{2s}{s}$ and $\binom{N-1}{t-s-1}$ from above and from below, more precisely, first we prove that

$$\frac{C_1 2^{2s}}{2^N \sqrt{N}} \leq \frac{s \binom{2s}{s}}{N \binom{N-1}{t-s-1}} \leq \frac{2^{2s}}{\sqrt{N}} \quad (2.12)$$

where $C_1 = e^{-\frac{1}{6}} \sqrt{\frac{2}{\pi}}$. In order to estimate the binomial coefficient $\binom{2s}{s}$ we use Stirling's formula

$$s! = s^s e^{-s} \sqrt{2\pi s} e^{\lambda_s},$$

where

$$\frac{1}{12s+1} \leq \lambda_s \leq \frac{1}{12s}$$

We may write

$$\binom{2s}{s} = \frac{2^{2s}}{\sqrt{\pi s}} e^{\lambda_{2s} - 2\lambda_s} \quad (2.13)$$

Taking into account the following inequalities for the errors

$$-\frac{1}{6s} \leq \lambda_{2s} - 2\lambda_s \leq 0$$

we get estimates from above and from below for $s \binom{2s}{s}$

$$\frac{e^{-\frac{1}{6}}}{\sqrt{\pi}} 2^{2s} \leq s \binom{2s}{s} \leq 2^{2s} \sqrt{s} \quad (2.14)$$

For the binomial coefficient $\binom{N-1}{t-s-1}$ we use the fact that its maximum is assumed at $t-s-1 = \lfloor \frac{N-2}{2} \rfloor$. Hence

$$1 \leq \binom{N-1}{t-s-1} \leq \binom{N-1}{\lfloor \frac{N-2}{2} \rfloor}$$

We can assume for simplicity that N is even. The case of N odd can be treated with little more effort. Then, as before, we use Stirling's formula

$$\begin{aligned} N \binom{N-1}{\frac{N}{2}-1} &= \frac{N!}{(\frac{N}{2}-1)! (\frac{N}{2})!} = \\ \frac{N}{2} \frac{N!}{(\frac{N}{2})!^2} &\leq \frac{N}{2} \frac{2^N}{\sqrt{\frac{N}{2}}} = 2^N \sqrt{\frac{N}{2}} \end{aligned}$$

Thus we obtain

$$N \leq N \binom{N-1}{t-s-1} \leq N \binom{N-1}{\lfloor \frac{N-2}{2} \rfloor} \leq 2^N \sqrt{\frac{N}{2}},$$

that yields with (2.14) the formula (2.12). Therefore we are able to estimate the mean $\frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}$ from below and above using the same series

$$\begin{aligned} 1 + \frac{C_1}{2^N \sqrt{N}} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \binom{N}{2s} \binom{N-2s}{t-2s} (u+v)^s \left(\frac{w}{2}\right)^{t-2s} \leq \\ \frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}(u, v, w) \leq \end{aligned}$$

$$1 + \frac{1}{\sqrt{N}} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \binom{N}{2s} \binom{N-2s}{t-2s} (u+v)^s \left(\frac{w}{2}\right)^{t-2s} \quad (2.15)$$

It remains to evaluate the series

$$\sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \binom{N}{2s} \binom{N-2s}{t-2s} (u+v)^s \left(\frac{w}{2}\right)^{t-2s} \quad (2.16)$$

We then state the next Lemma

Lemma 2.2 *The following equality*

$$\begin{aligned} & \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \binom{N}{2s} \binom{N-2s}{t-2s} (u+v)^s \left(\frac{w}{2}\right)^{t-2s} = \\ & \frac{1}{2} \left[\left(\sqrt{u+v} + \frac{w}{2} + 1 \right)^N + \left(-\sqrt{u+v} + \frac{w}{2} + 1 \right)^N \right] - 1 \end{aligned} \quad (2.17)$$

holds, for any N .

Proof of Lemma 2.2: Consider the sum (2.16) for s and t starting from 0 and denote it by $*$. Note that in $*$ there is an extra term at $t = s = 0$ which equals 1. In this case we can apply the following combinatorial equality (see [2])

$$\sum_{\nu} \binom{n}{\nu} \binom{n-\nu}{k-\nu} t^{\nu} = \binom{n}{k} (t+1)^k, \quad (2.18)$$

where the sum is over any ν such that $0 \leq \nu \leq k \leq n$. The summation with respect to s in $*$ coincides with the sum (2.18) restricted to the even indices ν . Owing to this fact we rewrite the sum over s in a more suitable form:

$$\begin{aligned} & \sum_{s=0}^{\lfloor \frac{t}{2} \rfloor} \binom{N}{2s} \binom{N-2s}{t-2s} (\sqrt{u+v})^{2s} \left(\frac{w}{2}\right)^{t-2s} = \\ & \sum_{s=0}^t \frac{1 + (-1)^s}{2} \binom{N}{s} \binom{N-s}{t-s} (\sqrt{u+v})^s \left(\frac{w}{2}\right)^{t-s} = \\ & \frac{1}{2} \sum_{s=0}^t \binom{N}{s} \binom{N-s}{t-s} \left(\frac{2\sqrt{u+v}}{w}\right)^s \left(\frac{w}{2}\right)^t + \end{aligned}$$

$$\frac{1}{2} \sum_{s=0}^t \binom{N}{s} \binom{N-s}{t-s} \left(-\frac{2\sqrt{u+v}}{w} \right)^s \left(\frac{w}{2} \right)^t$$

We apply now formula (2.18)

$$\begin{aligned} & \sum_{s=0}^{\lfloor \frac{t}{2} \rfloor} \binom{N}{2s} \binom{N-2s}{t-2s} (\sqrt{u+v})^{2s} \left(\frac{w}{2} \right)^{t-2s} = \\ & \frac{1}{2} \binom{N}{t} \left[\left(\frac{2\sqrt{u+v}}{w} + 1 \right)^t + \left(-\frac{2\sqrt{u+v}}{w} + 1 \right)^t \right] \left(\frac{w}{2} \right)^t = \\ & \frac{1}{2} \binom{N}{t} \left[\left(\sqrt{u+v} + \frac{w}{2} \right)^t + \left(-\sqrt{u+v} + \frac{w}{2} \right)^t \right] \end{aligned} \quad (2.19)$$

Summing the last term in (2.19) over t we get again a binomial formula

$$\begin{aligned} & \frac{1}{2} \sum_{t=0}^N \binom{N}{t} \left[\left(\sqrt{u+v} + \frac{w}{2} \right)^t + \left(-\sqrt{u+v} + \frac{w}{2} \right)^t \right] = \\ & \frac{1}{2} \left[\left(\sqrt{u+v} + \frac{w}{2} + 1 \right)^N + \left(-\sqrt{u+v} + \frac{w}{2} + 1 \right)^N \right] \end{aligned}$$

The Lemma 2.2. is so proved. \square

Now we are able to finish the proof of the Theorem

Upper bound: Taking into account (2.15) and (2.17) we obtain an upper bound for $\frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}$

$$\begin{aligned} & \frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}(u, v, w) \leq \\ & 1 - \frac{1}{\sqrt{N}} + \frac{1}{2\sqrt{N}} \left(\sqrt{u+v} + \frac{w}{2} + 1 \right)^N \left[1 + \left(\frac{-\sqrt{u+v} + \frac{w}{2} + 1}{\sqrt{u+v} + \frac{w}{2} + 1} \right)^N \right] \end{aligned}$$

Since

$$\frac{-\sqrt{u+v} + \frac{w}{2} + 1}{\sqrt{u+v} + \frac{w}{2} + 1} \leq 1,$$

we have proved the upper estimate in (2.4).

Lower bound: Analogously, we obtain as lower bound for $\frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}$

$$1 - \frac{C_1}{2^N \sqrt{N}} + \frac{C_1}{2^{N+1} \sqrt{N}} \left[\left(\sqrt{u+v} + \frac{w}{2} + 1 \right)^N + \left(-\sqrt{u+v} + \frac{w}{2} + 1 \right)^N \right] =$$

$$\begin{aligned}
& 1 - \frac{C_1}{2^N \sqrt{N}} + \frac{C_1}{2\sqrt{N}} \left[\left(\frac{\sqrt{u+v}}{2} + \frac{w}{4} + \frac{1}{2} \right)^N + \left(-\frac{\sqrt{u+v}}{2} + \frac{w}{4} + \frac{1}{2} \right)^N \right] = \\
& 1 - \frac{C_1}{2^N \sqrt{N}} + \frac{C_1}{2\sqrt{N}} \left(\frac{\sqrt{u+v}}{2} + \frac{w}{4} + \frac{1}{2} \right)^N \left[1 + \left(\frac{-\sqrt{u+v} + \frac{w}{2} + 1}{\sqrt{u+v} + \frac{w}{2} + 1} \right)^N \right]
\end{aligned} \tag{2.20}$$

Therefore we get the lower bound (2.4), with the constant

$$C = \frac{C_1}{2} \quad \text{for } -\sqrt{u+v} + \frac{w}{2} + 1 \geq 0$$

$$C = \frac{C_1}{2} \left(1 + \frac{-\sqrt{u+v} + \frac{w}{2} + 1}{\sqrt{u+v} + \frac{w}{2} + 1} \right) = \frac{C_1}{2} \left(\frac{w+2}{\sqrt{u+v} + \frac{w}{2} + 1} \right),$$

for $-\sqrt{u+v} + \frac{w}{2} + 1 < 0$. The Theorem is so proved. \square

3 Numerics for coloured hard-dimers

In this section we provide some supplementary material regarding hard-dimer combinatorics. We consider two enumeration problems which are different from that in section 2. Numerical techniques are being employed and the results may indicate interesting directions for further study. In the first part we count the number of hard-dimers for each configuration ξ_N , N fixed. The so obtained statistics is then shown in histograms.

In order to explain the algorithm we consider a particular configuration as shown in Fig.2.

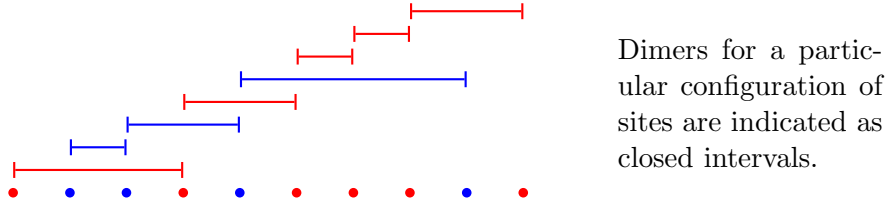


Figure 2: Dimers

Recall that a hard-dimer on a sequence of red and blue sites is a sequence of dimers which are non-overlapping. In order to count the number of hard-dimers on the fixed sequence ξ_N of red and blue sites, one can proceed as

follows:

Step 1. Generate a list of all dimers, i.e. codify their colour as well as their starting and end points. E.g. for the dimers in Fig.2 this is achieved by defining the set

$$ED :=$$

$$\{\{1, 4, r\}, \{2, 3, b\}, \{3, 5, b\}, \{4, 6, r\}, \{5, 9, b\}, \{6, 7, r\}, \{7, 8, r\}, \{8, 10, r\}\}$$

Step 2. For each $l \in \{1, \dots, |ED|\}$ let $K(l)$ be the set of subsets of ED containing l elements. Do the following for $l = 1$ to $l = |ED|$. Scan the elements of $K(l)$. If for an element in $K(l)$ the dimers are non-overlapping, increase the number of hard-dimers by one, otherwise keep it unchanged. At the end one has the number of hard-dimers for a particular configuration ξ_N .

For fixed N , let S_N be the set of all ξ_N and let $\xi_N(D)$ denote the number of hard-dimers for ξ_N . This number is obtained following the steps 1-2 above. Then let $X = (\xi_N(D))_{\xi_N \in S_N}$ be the sample obtained from this numeric counting procedure. The histograms for the standardized sample $Y = (X - \text{mean}(X))/\text{st.deviation}(X)$ are depicted in Fig.3 and Fig.4 below.

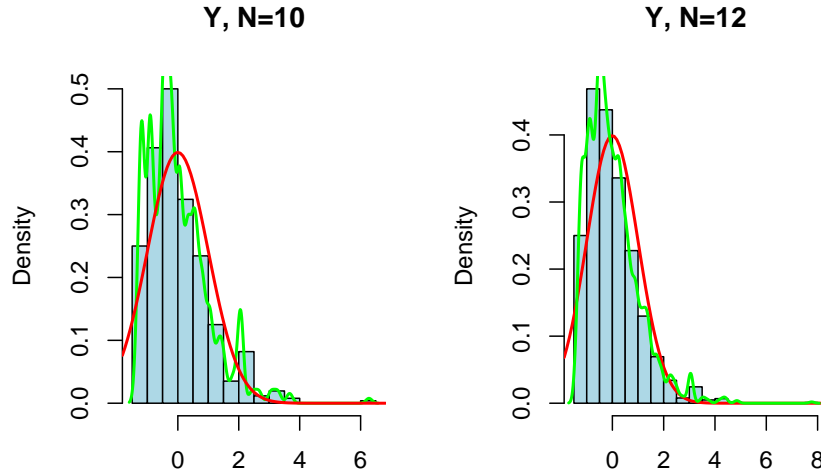
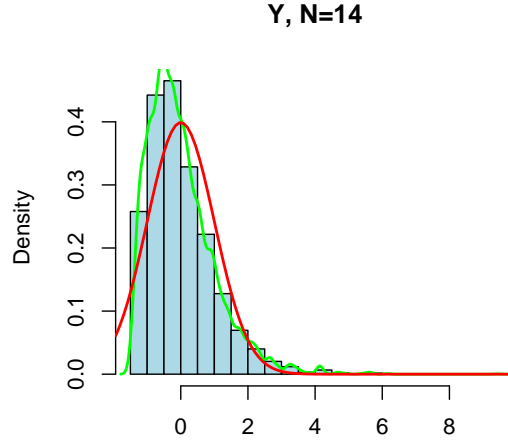


Figure 3: Histograms of Y

Figure 4: Histogram of Y

Remark 3.1 *The red curves in the histograms of Y above represent the standard normal distribution. On the other hand, the green curves are obtained by smoothing of Y with a Gaussian kernel having standard deviation 0.1. With increasing N the green curves become less oscillating and a tendency towards a definite distribution becomes visible. Unfortunately, the algorithm becomes very time consuming for N larger 14 and it is not clear whether the distributions tend to a standard normal distribution.*

In this second part we generate hard-dimers randomly. Roughly speaking, this is achieved by first choosing N red or blue sites randomly, each with probability one half and then for each such configuration dimers are generated. Here again randomness enters in that one dimer and the next one are separated by some random choice. The algorithm can be implemented in the following steps:

Step 1. Generate a sequence of red and blue sites. The colours appear with probability one half.

Step 2. For the first site, depending on the colour, we choose randomly the base elements with attributes equal or right, see Fig.5. For example, if the colour is blue, br or be are picked at random with probability one half.

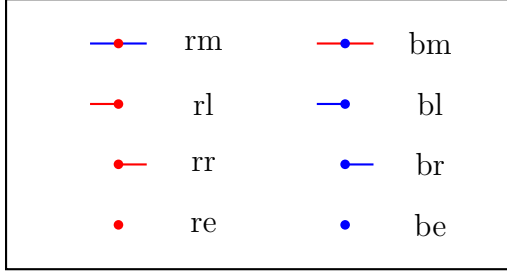


Figure 5: Base elements

These are the building elements from which the random hard-dimers are constructed.

The acronyms mean: $re \doteq$ red-equal, $rr \doteq$ red-right, $rl \doteq$ red-left, $rm \doteq$ red-mixed, and similarly for blue.

Step 3. Suppose we have generated the configuration up to the k^{th} site and want to configure the $(k+1)^{\text{th}}$ site. Here we have to distinguish the four possible combinations of the two colours. Suppose the colours are both blue. Then, given the element at the k^{th} site we have to choose the base element of the $(k+1)^{\text{th}}$ site according to the following table:

k^{th} site is blue	$(k+1)^{\text{th}}$ site is blue
br	bl
bm	bm
be	random choice $\in \{be, br\}$
bl	random choice $\in \{be, br\}$

In the case of a blue and a red site we take:

k^{th} site is blue	$(k+1)^{\text{th}}$ site is red
br	rm
bm	rd
be	random choice $\in \{re, rr\}$
bl	random choice $\in \{re, rr\}$

The random choice happens with probability one half. Similarly for the other two cases, taking into account the properties of dimers.

Step 4. For the last site the elements are chosen from the base elements with attributes left and equal only. If this is not possible, then a \times is being inserted. In this way we indicate that the dimer remains unfinished and will therefore not be counted in the last step 5, see Fig.6. Concretely, we follow the rules:

penultimate site is blue	last site is blue
br	bl
bm	x
be	be
bl	be

and

penultimate site is blue	last site is red
br	x
bm	rl
be	re
bl	re

Similarly for the other two cases.

Step 5. Count the number of dimers.



Figure 6: A hard-dimer being omitted

Now, let S_N^r denote a random sample of a certain size $|S_N^r| = m$ which is obtained by repeating the above procedure, steps 1-4, m times. Again, N is given by the number of sites. For $\eta \in S_N^r$ let $\eta(d)$ denote the number of dimers counted in η . Then in Fig.7 the histograms are shown for two samples $X = (\eta(d))_{\eta \in S_N^r}$.

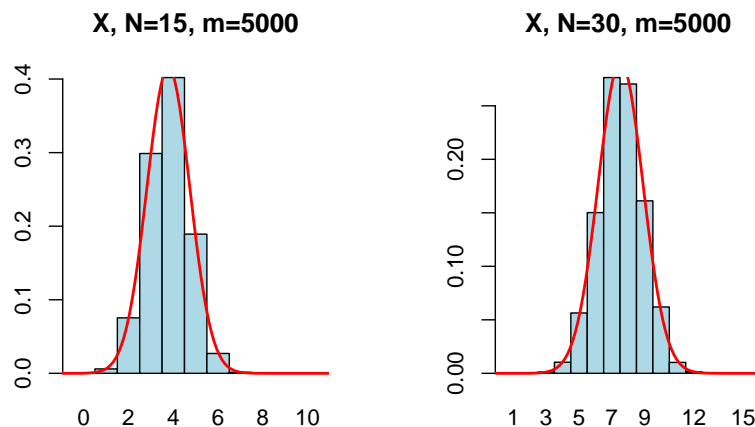


Figure 7: Histograms of X

Remark 3.2 *In the histograms for X above, the red curves have been chosen to have the same mean and standard deviation as the corresponding histograms. The very good approximation by the histograms makes a central limit property visible.*

References

- [1] Benedetti, D., Loll, R., Zamponi, F.: $(2+1)$ -dimensional quantum gravity as the continuum limit of causal dynamical triangulations. *Phys. Rev. D* 76 (2007), no. 10, 104022, 26 pp.
- [2] Feller, W.: *An introduction to probability theory and its applications*. Vol. I. Third edition John Wiley & Sons, Inc., New York-London-Sydney 1968